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Integration Rules of the Second Kind

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Let $-\infty < a < b < \infty$ and let f be a real function defined on $[a, b]$ with

$$-\infty < \lambda = \inf_{a \leqslant x \leqslant b} f(x) \leqslant \Lambda = \sup_{a \leqslant x \leqslant b} f(x) < \infty.$$

Assume first that $a \geqslant 0$, $\lambda \geqslant 0$, and that f is continuous and strictly increasing on $[a, b]$. Then the geometry involved immediately implies that

$$\int_a^b f(x)\, dx = \Lambda(b - a) - \int_\lambda^\Lambda [f^{-1}(y) - a]\, dy. \qquad (1)$$

In particular, one can approximate $\int_a^b f(x)\, dx$ by applying an integration rule to the right-hand side of (1). This procedure can be called an integration rule of the "second kind" (for f, a and b).

EXAMPLE 1. "Simpson's rule of the second kind" for f, a and b is

$$\int_a^b f(x)\, dx \sim \Lambda b - \lambda a - \frac{\Lambda - \lambda}{6}\left[a + 4f^{-1}\left(\frac{\lambda + \Lambda}{2}\right) + b\right]. \qquad (2)$$

If $f(x) \equiv x^{1/3}$, $a = 0$, $b = 1$, then the right-hand side of (2) is $\frac{3}{4}$, the *exact* value of $\int_a^b f(x)\, dx$, while applying Simpson's rule itself to f, a and b gives only the crude estimate 0.6958... .

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Consider a standard integration rule with an error term involving the derivative of some order of the integrand. In the corresponding rule of the second kind, there will generally appear a derivative of the same order, but of the function inverse to the integrand. This can be evaluated either directly, by successive differentiations, or by using the explicit formula for the nth derivative of an inverse function [1, pp. 738–740; 3, pp. 20–22, 290–293].

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We can write (1) as

$$\int_a^b f(x)\,dx = \Lambda(b-a) - \int_\lambda^\Lambda e(y)\,dy, \tag{3}$$

where, for every $y \in [\lambda, \Lambda]$, $e(y)$ is the (Lebesgue) measure of the set of all $x \in [a, b]$ for which $f(x) \leqslant y$.

Now drop the assumptions of the second sentence of this note, but assume that f is measurable in $[a, b]$. Then with the above definition of $e(y)$, $y \in [\lambda, \Lambda]$, (3) still holds [2, p. 262; the setting there is slightly different from ours]. The integral on the right of (3) continues to be a Riemann integral, showing that the Lebesgue integral of a bounded measurable real function over a finite interval can be represented by a Riemann integral. If $\lambda \geqslant 0$, (3) has a clear geometric interpretation.

Again, to compute approximately $\int_a^b f(x)\,dx$, we may apply to it an integration rule of the second kind, i.e., apply an ordinary integration rule to the right-hand side of (3). The above example shows that we may thus get a better result than by applying the rule directly to the left-hand side. In any case, by having ready in his mind such a procedure, a numerical analyst engaged in numerical integration will enrich his kit of tools. This procedure can be programmed in various situations, e.g., when what is given of f is a discrete table of values.

EXAMPLE 2. If $f(x) = \sin x$, $a = 0$, $b = \pi$, then Simpson's rule for $\int_a^b f(x)\,dx$ gives $2\pi/3$, an error of 4.7 ... %, while Simpson's rule of the second kind gives $11\pi/18$, an error of 4.0 ... %.

EXAMPLE 3. Suppose f is continuous in its domain, $[a, b]$, f assumes no value infinitely often, and λ and Λ ($>\lambda$) are known. Let n be an even integer $\geqslant 2$, and set $y_k = \lambda + (k/n)(\Lambda - \lambda)$, $k = 0, 1, \ldots, n$. Assume that, for every $t \in [a, b]$, $f(t)$ measures some quantity arising in engineering, physics or economics, at time t. However, we are unable to determine $f(t)$ for arbitrary t. The arrangement is rather that, in the time interval $a \leqslant t \leqslant b$, whenever the quantity in question assumes one of the values y_k, this fact as well as the time of occurrence are recorded. Suppose also that, whenever two consecutive recordings arise from the same y_k, a record is made as to whether, in the time interval in between, $f(t)$ was smaller than y_k or larger. Assume likewise that, if the very first recording is at time $t' > a$, a record is made as to whether $f(t)$ was smaller or larger than $f(t')$ for $t < t'$, and, if the very last recording is at time $t'' < b$, a record is made of whether $f(t)$ was smaller or larger than $f(t'')$ for $t > t''$. Then we get at once all values $e(y_k)$, $k = 0, 1, \ldots, n$, and hence can approximate $\int_a^b f(x)\,dx$ by using a compound Simpson's rule of the second kind.

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